Hydrodynamics of smectic-*C* liquid crystals: Field and flow induced instabilities in confined geometries

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Following the Ericksen-Leslie approach, we formulate a complete nonlinear macroscopic theory of the isothermal hydrodynamics of smectic-*C* liquid crystals. We assume an asymmetric stress tensor and incorporate the essential features of a hydrodynamic theory of a smectic phase, i.e., permeation and variations in layer spacing. Using Onsager's reciprocity relations, we find that entropy production is described by 16 viscosity coefficients and a permeation constant associated with the dissipative dynamics of the layered system. We study the reorientation dynamics of the **c** vector under the destabilizing influence of an external field. We stress that permeation is important and that transverse flows along and normal to the layers exist. We have also studied certain instabilities that can arise in shear flows. As a consequence of permeation, in Poiseuille flow with the layers parallel to the plates, we find that the length of the inlet section can be very large being many times the lateral dimension. When the layers are perpendicular to the plates, an analog of the nematic Hall effect is shown to exist even in the absence of an aligning external field. [S1063-651X(98)03305-4]

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I. INTRODUCTION

The hydrodynamics of smectic-*C* (Sm-*C*) liquid crystals (LCs) is different because it incorporates the flow of a fluid as in ordinary fluid mechanics, the dynamics of oriented media as in nematic LCs, and also the dynamics of layered media such as smectic-*A* LCs. As a consequence of the coupling that exists between these, even in the linear regime, the theory is complex and theoretical investigations on macroscopic flows have been very limited. In 1972, Martin, Parodi, and Pershan (MPP) [1] formulated a linearized hydrodynamic theory of Sm-*C* LCs that has been very successful in the study of fluctuations. A more complete nonlinear version of this theory has been developed by others [2,3] and the results follow along the lines of MPP. The stress tensor is assumed to be *symmetric* and the phenomenon of *permeation* is also described when there exists flow in a direction normal to the layers. However, these theories have not been used to describe macroscopic flows. Nearly 20 years later, Leslie, Stewart, and Nakagawa (LSN) [4,5], following the approach of Ericksen and Leslie [6,7] in the formulation of the hydrodynamics of uniaxial nematic LCs, developed a theory that allows for nonlinear coupling between curvature of the layers, director orientation, and macroscopic flow. They use an *asymmetric* stress tensor, but permeation and layer dilatation or compression are not included in their theory. Hence these two approaches are different. The theory of MPP has been used to describe fluctuations and the theory of LSN has been employed recently in describing reorientation dynamics and the effects of ''backflow'' [8–11].

Here we formulate a macroscopic hydrodynamic theory based on the principles of classical mechanics that generalize concepts of linear and angular momentum employed in classical Newtonian mechanics. We assume an *asymmetric* stress tensor and incorporate the essential features of a hydrodynamic theory of a smectic phase, that is, *permeation* and *variations in layer spacing*. First, we present an outline of the complete nonlinear macroscopic theory of the isothermal hydrodynamics of Sm-*C* LCs that can be applied even to chiral smectic-*C* (Sm-*C**) LCs. Our main aim in proposing this theory is to systematically derive the equations of motion by generalizing forces and torques. Using Onsager's reciprocity relations, we find that the entropy production in a compressible system of monoclinic symmetry embodying dissipative torques is described by 16 viscosity coefficients and a permeation constant arising from the dissipative dynamics of the layered system. It may be mentioned in passing that LSN's theory has 20 viscosity coeffients for the incompressible case. Our theory incorporates coupling between the different hydrodynamic variables, viz., velocity, layer spacing, and the **c** vector. It is a covariant description in the sense that the constraints of the system have been incorporated and the theory is not limited to mere perturbations of planar structures. The main problem regarding experimental and theoretical investigations of Sm-*C* LCs has been the complexity arising from such couplings. Focal conics, chevron textures, and other topological defects that hamper experimental work require the complete nonlinear theory for their desciption. We develop a linear analysis of instabilities that bring out the intimate coupling between the hydrodynamic variables. The onset of such instabilities might lead to the generation of the above-mentioned topological defects. To reduce the complexity of the problems, the earlier studies [5,8–10] assumed that (i) the layers are flat and even introduced external counter torques to ensure such a condition and (ii) the flow along the layer normal is negligible, thus neglecting permeation. Though these assumptions appear reasonable, they lead to mathematical inconsistencies and an incomplete description of even well-known instabilities such as the Fréedericksz transition. We have studied certain field

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and flow induced instabilities and our analysis leads to a more complex behavior in each case. Incidentally, even though we work with an asymmetric stress tensor, the linearized version of our theory is similar to that of MPP (as in the case of uniaxial nematics LCs) apart from a difference in the choice of forces and fluxes in developing the constitutive equations. This choice merely depends on the type of the problem one is tackling. For instance, the Ericksen-Leslie approach in uniaxial nematics LCs is instructive when one considers field and flow induced instabilities; however, MPP's theory is useful to study only small fluctuations and wave propagation [12].

The reorientation dynamics of the director under the destabilizing influence of an external field is of paramount importance in modeling electro-optic devices. Recent works [8–10] emphasize the effects of backflow and point out that transverse flow within the fluid layer alters the director profile. Barratt and Duffy [11] reanalyzed this problem and showed that the earlier analyses lead to overdetermined equations. These authors included flow in the direction normal to the layers, but without incorporating permeation. Qualitatively, all these investigations result in predicting similar behavior. Since macroscopic flow normal to the layers involves the effects of permeation, the real dynamics is different. We present preliminary investigations on macroscopic dynamics involving permeation. We point out that transverse flows and flow normal to the layers that is coupled to the curvature elasticity of the layers are essential for a complete hydrodynamic desciption of smectic LCs. We have also analyzed certain instabilities that could arise in shear flows. Finally, we study Poiseuille flow in two geometries, viz., with the layers parallel and perpendicular to the bounding plates. In general, in the standard discussion of Poiseuille flow, we do not consider the process by which the fluid attains a steady state. This is invariably confined to a small region called the inlet section at the entry point. Surprisingly, in the flow of smectics LCs, we cannot ignore in certain geometries the length of the inlet section. We find that in the case of flow with layers slipping past each other, the length of the inlet section is very large as a consequence of permeation in the region of nonsteady flow. This result is true for all smectics LCs in general. In the case of Poiseuille flow within layers, an analog of the nematic Hall effect, i.e., a pressure gradient in a direction perpendicular to the velocity and the velocity gradient, exists even in the absence of an external field. This is peculiar to Sm-*C* LCs and does not exist in Sm-*A* LCs.

II. HYDRODYNAMICS OF SMECTIC-*C* LCs

A Sm-*C* liquid crystal is a layered structure in which the director $\mathbf{n}$, which represents the preferred direction of the local molecular orientation, is everywhere inclined at the same angle with respect to the layer normal. These liquid crystals can be described by a density variation along the layer normal,

$$\rho=\rho_0+[\delta\rho \exp iq_S\psi+\text{c.c.}], \tag{1}$$

where $q_S=2\pi/d$, with d the layer spacing and $\delta\rho$ is the amplitude of the density variations. The smectic layers are surfaces with $\psi(\mathbf{r})=\text{const}$. A deformed smectic stucture, with the Z axis normal to the undeformed layers, is described by $\psi(\mathbf{r})=z-u(\mathbf{r})$.

The hydrodynamic equations are always in terms of the hydrodynamic variables. These variables are characterized by slow decay times proportional to some power of their wavelengths. The number of such variables is determined by the sum of the number of conservation laws and the number of ''continuous broken symmetries.'' As in ordinary fluids, here also we have the density, the components of the velocity, and the energy density as five hydrodynamic variables associated respectively with the conservation of the mass, the momentum, and the energy. In addition to these, we have variables describing the broken translational invariance in one dimension, as described by the layering, and the broken rotational invariance of the director $\mathbf{n}$, as described by a transverse twofold axis. These variables are, respectively, the scalar variable ψ and the vector $\mathbf{c}$ that is the projection of $\mathbf{n}$ onto the layers. The vector $\mathbf{c}$ is subject to the constraints $\mathbf{c}\cdot\mathbf{c}=1$ and $\mathbf{c}\cdot\mathbf{N}=0$, where $\mathbf{N}$ is the unit layer normal given by $\nabla\psi/|\nabla\psi|$. It may be remarked here that in the theory of LSN, the constraint $\nabla\times\mathbf{N}=0$ is imposed on the structural deformations. This by itself cannot explicitly take care of general layer deformations and the process of permeation. On the other hand, our constraints on $\mathbf{c}$ and $\mathbf{N}$ allow for these processes. It should be noted that in this hydrodynamical description of Sm-*C* LCs, a variation in the tilt of $\mathbf{n}$ with respect to the layer normal is not permitted. The macroscopic dynamics of any system does not depend on whether the stress tensor is symmetric or asymmetric. We choose an asymmetric stress tensor since it is more appropriate to these systems that allow for internal torques.

A. Conservation laws

In writing down the complete isothermal hydrodynamics of Sm-*C* LCs, we consider a compressible material at each point x_k and also assume that the system has reached statistical equilibrium locally. The orientation of the director $\mathbf{n}$ is completely described by $\mathbf{c}$ and $\mathbf{N}$ since the tilt of $\mathbf{n}$ with respect to $\mathbf{N}$ is nothing but the order parameter (tilt angle) of the Sm-*C* phase. The approach of LSN has been to consider Sm-*C* LCs as a biaxial system described by $\mathbf{c}$ and $\mathbf{N}$. Then torques on the system can be considered to consist of two parts [13]. One part arises from the familiar moment of linear momentum of the fluid particle, while the other part is due to torques that change $\mathbf{c}$ and $\mathbf{N}$. Since $\mathbf{N}$ is completey described by $\nabla\psi/|\nabla\psi|$, generalized torques leading to changes in $\mathbf{N}$ can be equivalently described by generalized forces resulting in second-order gradients in ψ. An example of this is in the equations of equilibrium of smectic-*A* LCs. Here, with ψ as the hydrodynamic variable, the stress tensor does not need to have an antisymmetric part because torques are totally due to moment of linear momentum of fluid particles and the conservation of angular momentum is implicit. However, in Sm-*C* LCs there is an additional internal degree of freedom due to $\mathbf{c}$. The difference between the total angular momentum and the moment of linear momentum can be attributed to an internal angular momentum leading to changes in the $\mathbf{c}$ vector. In Sm-*C* LCs, the equation for the conservation of this angular momentum acquires an independent status. This leads, in Sm-*C* LCs, to an equation that is similar to the Oseen equation of nematic LCs.

At any time t, the conservation (for a volume V of the fluid bounded by surface A) of mass, linear momentum, energy, and angular momentum, respectively, leads to the equations

$$\frac{D}{Dt}\int_V \rho\, dV=0, \tag{2}$$

$$\frac{D}{Dt}\int_V \rho v_i dV=\int_V \rho f_i dV+\oint_A \sigma_{ji} dA_j, \tag{3}$$

$$\frac{D}{Dt}\int_V \rho\, dV[\tfrac{1}{2}v_i v_i+U+\tfrac{1}{2}\rho_c \dot{c}_i \dot{c}_i]=\int_V \rho\, dV[f_i v_i+G_i \dot{c}_i] + \oint_A dA_j[\sigma_{ji}v_i+\pi_{ji}\dot{c}_i], \tag{4}$$

$$\frac{D}{Dt}\int_V \rho\, dV e_{ijk}[x_j v_k+\rho_c c_j \dot{c}_k]=\int_V \rho\, dV\, e_{ijk}[x_j f_k+c_j G_k] + \oint_A dA_l e_{ijk}[x_j \sigma_{lk} + c_j \pi_{lk}]. \tag{5}$$

Further, the Oseen equation for the **c** vector is

$$\frac{D}{Dt}\int_V \rho\rho_c \dot{c}_i dV=\int_V [\rho G_i+g_i]dV+\oint_A \pi_{ji} dA_j, \tag{6}$$

where ρ is the density, D/Dt is the material derivative, dA_i is a vector representing the area element, and dV is the volume element. Further, $v_i, f_i, \sigma_{ji}, G_i, g_i$, and π_{ji} are, respectively, the components of the velocity, force, stress tensor, external director body force, intrinsic director body force, and director surface stress tensor. In Eq. (4), U is the internal energy per unit mass, and ρ_c is the moment of inertia per unit mass. This moment of inertia depends only on the degree of orientational order and hence is a constant at a given temperature and pressure.

With Reynold's transport theorem and Gauss's theorem, Eqs. (2)–(6) imply

$$\frac{D\rho}{Dt}+\rho v_{k,k}=0, \tag{7}$$

$$\rho\frac{Dv_i}{Dt}=\rho f_i+\sigma_{ji,j}, \tag{8}$$

$$\rho\rho_c\frac{D^2 c_i}{Dt^2}=\rho G_i+g_i+\pi_{ji,j}, \tag{9}$$

$$\rho\frac{DU}{Dt}=\sigma_{ji}d_{ij}+\pi_{ji}(\dot{c}_{i,j}-\omega_{ik}c_{k,j})-g_i(\dot{c}_i-\omega_{ik}c_k), \tag{10}$$

$$\sigma_{jk}+c_{j,l}\pi_{lk}-c_j g_k=\sigma_{kj}+c_{k,l}\pi_{lj}-c_k g_j, \tag{11}$$

where $d_{ij}=\frac{1}{2}(v_{i,j}+v_{j,i})$, $\omega_{ij}=\frac{1}{2}(v_{i,j}-v_{j,i})$, $\sigma_{ji,j}=\partial\sigma_{ji}/\partial x_j$, $\pi_{ji,j}=\partial\pi_{ji}/\partial x_j$, $v_{i,j}=\partial v_i/\partial x_j$, $c_{i,j}=\partial c_i/\partial x_j$, and $\dot{c}_i=Dc_i/Dt$. The moment of inertia associated with the **c** vector, ρ_c, is an extremely small quantity and is negilgible. In other words, we ignore this inertia term in all our future discussion.

If S is the entropy per unit mass at an absolute temperature T, then the Helmholtz free energy per unit mass is $F=U-TS$. According to thermodynamics, $\dot{S}$, the rate of entropy generation, is always positive. At constant temperature, this means

$$\rho T\dot{S}=\rho\dot{U}-\rho\dot{F}\geqslant 0. \tag{12}$$

B. Constitutive equations

We assume that F is a single-valued function of the variables $\rho, \psi_{,i}, N_i, N_{i,j}, c_i$, and $c_{i,j}$. Using the chain rule of differential calculus to expand $\dot{F}$ and making use of Langrange multipliers to express the constraints $\mathbf{c}\cdot\mathbf{c}=1$, $\mathbf{c}\cdot\mathbf{N}=0$, and the fact that $\mathbf{N}=\nabla\psi/|\nabla\psi|$, we have

$$\dot{F}=\frac{\partial F}{\partial\rho}\frac{D\rho}{Dt}+\frac{\partial F}{\partial\psi_{,i}}\frac{D\psi_{,i}}{Dt}+\frac{\partial F}{\partial N_i}\frac{DN_i}{Dt}+\frac{\partial F}{\partial N_{i,j}}\frac{DN_{i,j}}{Dt} +\frac{\partial F}{\partial c_i}\frac{Dc_i}{Dt}+\frac{\partial F}{\partial c_{i,j}}\frac{Dc_{i,j}}{Dt}+\lambda N_i\frac{Dc_i}{Dt}+\lambda c_i\frac{DN_i}{Dt} +\gamma c_i\frac{Dc_i}{Dt}+\beta_i\frac{D}{Dt}(\psi_{,i}-|\psi_{,i}|N_i),$$

where λ, γ, and β_i are the Langrange multipliers. Then it is easy to show that

$$\int_V \rho\dot{F}\, dV=\oint_A dA_i\left[\Lambda_i\dot{\psi}+\rho\frac{\partial F}{\partial N_{j,i}}\dot{N}_j+\rho\frac{\partial F}{\partial c_{j,i}}\dot{c}_j\right] +\int_V dV\left[d_{ij}\left(-\rho^2\frac{\partial F}{\partial\rho}\delta_{ij}-\psi_{,i}\Lambda_j-\rho N_{k,i}\frac{\partial F}{\partial N_{k,j}} -\rho c_{k,i}\frac{\partial F}{\partial c_{k,j}}\right)-\omega_{ij}\left(\psi_{,i}\Lambda_j+\rho N_{k,i}\frac{\partial F}{\partial N_{k,j}}-c_j h_i^c +\rho c_{k,i}\frac{\partial F}{\partial c_{k,j}}-\lambda\rho N_i c_j\right)+\dot{N}_i(h_i^N+\lambda\rho c_i -\beta_i\rho|\psi_{,m}|)-\dot{\psi}\Lambda_{i,i}+(\dot{c}_i-\omega_{ij}c_j)(h_i^c+\lambda\rho N_i +\gamma\rho c_i)\right],$$

where $|\psi_{,i}|$ is a measure of layer curvature and dilatation or contraction, $\Lambda_i=\rho(\partial F/\partial\psi_{,i})+\rho\beta_j P_{ji}$, $h_i^N=\rho(\partial F/\partial N_i)-[\rho(\partial F/\partial N_{i,j})]_{,j}$, $h_i^c=\rho(\partial F/\partial c_i)-[\rho(\partial F/\partial c_{i,j})]_{,j}$, and $P_{ij}=\delta_{ij}-N_i N_j$. With $\pi_{ji}=\rho(\partial F/\partial c_{i,j})$, $N_i h_i^c=-\rho\lambda$, $N_i g_i=-\rho N_i(\partial F/\partial c_i)$, $P_{ik}(g_k+\pi_{lk,l}+h_k^c)+\rho\gamma c_i=P_{ik}g'_k$, $\sigma_{ji}=\sigma'_{ji}-\rho^2(\partial F/\partial\rho)\,\delta_{ij}-\rho\psi_{,i}(\partial F/\partial\psi_{,j})-N_i P_{kj}h_k^N-\rho N_{k,i}(\partial F/\partial N_{k,j})-\rho c_{k,i}(\partial F/\partial c_{k,j})$, and making use of $\dot{N}_i=(P_{ik}/|\psi_{,m}|)(D\psi_{,k}/Dt)$, the expression for entropy generation is

$$T\int_V \rho\dot{S}\, dV = \oint_A dA_i\left[-\left(\rho\frac{\partial F}{\partial\psi_{i,i}}+\frac{P_{ki}}{|\psi_{,m}|}h_k^N\right)\dot{\psi}-\rho\frac{\partial F}{\partial N_{j,i}}\dot{N}_j\right] + \int_V dV\left[\sigma'_{ji}d_{ij}+\omega_{ij}W_{ij}+\dot{\psi}\left(\rho\frac{\partial F}{\partial\psi_{,i}} + \frac{P_{ki}h_k^N}{|\psi_{,m}|}\right)_{,i} - P_{ik}g'_k(\dot{c}_i-\omega_{ij}c_j)\right], \tag{13}$$

where

$$W_{ij}=\rho\psi_{,i}\frac{\partial F}{\partial\psi_{,j}}+\rho N_{k,i}\frac{\partial F}{\partial N_{k,j}}-\rho c_j\frac{\partial F}{\partial c_i}-\rho c_{j,k}\frac{\partial F}{\partial c_{i,k}} + \rho c_{k,i}\frac{\partial F}{\partial c_{k,j}}+N_iP_{kj}h_k^N.$$

Since ω_{ij} can be varied arbitrarily by superposed rigid rotations, $W_{ij}=W_{ji}$. Hence, from Eq. (11) we get the relation

$$\sigma'_{ji}-P_{ik}c_jg'_k=\sigma'_{ij}-P_{jk}c_ig'_k. \tag{14}$$

At this point, we recapitulate the interpretation of Kléman and Parodi [14] regarding the surface terms that appear in Eq. (13). With layers parallel to a plane surface, the first term describes the effects due to an imposition of a dilatation (or contraction) of layers. When the dilatation is large, this process leads to the creation of edge dislocations and then the number of layers in the system will not be conserved. When the layers are normal to a plane surface, the second term leads to depinning of layers along the surface in order to relax their bending. The last term plays the role of a surface torque. We shall not deal with these surface terms any further since we are interested in problems wherein the number of layers is conserved and the surface torques and layer depinning are considered to be absent due to boundary conditions. Weinan [15], following Kléman and Parodi, has very recently developed a nonlinear continuum theory of smectic-*A* liquid crystals. His derivation and interpretation of the permeative force is similar to that presented in this paper.

The entropy generation in the bulk satisfies the inequality

$$\rho T\dot{S}=\sigma'_{ji}d_{ij}+\dot{\psi}\left(\rho\frac{\partial F}{\partial\psi_{,i}}+\frac{P_{ki}h_k^N}{|\psi_{,m}|}\right)_{,i} - P_{ik}g'_k(\dot{c}_i-\omega_{ij}c_j)\geqslant 0. \tag{15}$$

The first term is associated with shape and volume changes as in normal hydrodynamics, the second describes dissipative dynamics of the layered structure, and the third is due to dissipative torques along the layer normal and acting on the **c** vector. It should be noted that torques on the **c** vector that distort the layer are described by the second term and we have removed such contributions from the last term (not to count the same thing twice, of course) by making use of the constraint $\mathbf{c}\cdot\mathbf{N}=0$.

C. Free energy and dissipative part of the stress tensor

The free-energy density and the dissipative part of the stress tensor should be invariant under the transformations: $\mathbf{N}\to-\mathbf{N}$ and $\mathbf{c}\to-\mathbf{c}$. The free energy density $F'=\rho F$ of Sm-*C* satisfying this constraint has been worked out by Nakagawa [16]. It has the form

$$\begin{aligned}\rho F=&\frac{A_{11}}{2}(\mathbf{c}\cdot\nabla\mathbf{a}\cdot\mathbf{b})^2+\frac{A_{21}}{2}(\mathbf{b}\cdot\nabla\mathbf{a}\cdot\mathbf{b})^2+\frac{A_{12}}{2}(\mathbf{c}\cdot\nabla\mathbf{a}\cdot\mathbf{c})^2\\ &+\frac{L_1}{2}(\mathbf{c}\cdot\nabla\mathbf{a}\cdot\mathbf{N})^2+\frac{L_2}{2}(\mathbf{b}\cdot\nabla\mathbf{a}\cdot\mathbf{N})^2+\frac{L_3}{2}(\mathbf{N}\cdot\nabla\mathbf{a}\cdot\mathbf{N})^2\\ &+L_{13}(\mathbf{c}\cdot\nabla\mathbf{a}\cdot\mathbf{N})(\mathbf{N}\cdot\nabla\mathbf{a}\cdot\mathbf{N})+\frac{B}{2}(\mathbf{a}\cdot\mathbf{N}-1)^2\\ &+\frac{B_1}{2}(\mathbf{c}\cdot\nabla\mathbf{c}\cdot\mathbf{b})^2+\frac{B_2}{2}(\mathbf{b}\cdot\nabla\mathbf{c}\cdot\mathbf{b})^2+\frac{B_3}{2}(\mathbf{N}\cdot\nabla\mathbf{c}\cdot\mathbf{b})^2\\ &+B_{13}(\mathbf{c}\cdot\nabla\mathbf{c}\cdot\mathbf{b})(\mathbf{N}\cdot\nabla\mathbf{c}\cdot\mathbf{b})-C_1(\mathbf{c}\cdot\nabla\mathbf{a}\cdot\mathbf{b})(\mathbf{c}\cdot\nabla\mathbf{c}\cdot\mathbf{b})\\ &-C_2(\mathbf{b}\cdot\nabla\mathbf{a}\cdot\mathbf{b})(\mathbf{b}\cdot\nabla\mathbf{c}\cdot\mathbf{b})-M_1(\mathbf{c}\cdot\nabla\mathbf{a}\cdot\mathbf{N})(\mathbf{b}\cdot\nabla\mathbf{a}\cdot\mathbf{b})\\ &+M_2(\mathbf{c}\cdot\nabla\mathbf{a}\cdot\mathbf{N})(\mathbf{c}\cdot\nabla\mathbf{a}\cdot\mathbf{c})+N_1(\mathbf{c}\cdot\nabla\mathbf{a}\cdot\mathbf{N})(\mathbf{b}\cdot\nabla\mathbf{c}\cdot\mathbf{b})\\ &+N_2(\mathbf{b}\cdot\nabla\mathbf{a}\cdot\mathbf{N})(\mathbf{N}\cdot\nabla\mathbf{c}\cdot\mathbf{b},)\end{aligned} \tag{16}$$

where $\mathbf{a}=\nabla\psi$ ($\mathbf{a}=\mathbf{N}$ when there is no layer dilatation or contraction) and $\mathbf{b}=\mathbf{N}\times\mathbf{c}$. When the variations are assumed to be small, this reduces to the free-energy density of the de Gennes and Prost [12]. It should be mentioned in passing that in this model of Nakagawa [16], instead of $\nabla\times\mathbf{N}=0$, it is $\nabla\times\mathbf{a}=0$. This condition also precludes the existence of dislocations and in addition allows for a more general description of layer distortions. We are adopting here the procedure of Nakagawa.

Then the entropy generation inequality (15) implies that

$$\dot{\psi}=\lambda_p\left(\rho\frac{\partial F}{\partial\psi_{,i}}+\frac{P_{ki}h_k^N}{|\psi_{,m}|}\right)_{,i}, \tag{17}$$

where λ_p is a constant called the permeation constant introduced by Helfrich [17] for layered systems. Equation (17) has the form of a continuity equation. Within the context of continuum mechanics, Eq. (17) describes a situation in which any relative motion of the fluid particles with respect to the layers results in a distortion of the layers. Further, as Helfrich pointed out [17], with the layers fixed at the boundaries, any flow normal to the layers would experience a large viscosity, and the velocity in that direction would be proportional to the pressure gradient just as in flow through porous media. With the layers fixed rigidly at the walls, a part of the pressure gradient distorts the layers and the remaining part leads to fluid flow through a ''fixed layered structure.'' The permeation constant λ_p can be roughly estimated to be $\lambda_p\eta\sim d^2$, where η and d are, respectively, the viscosity coefficient and the smectic layer separation. Roughly, $\lambda_p \sim 10^{-13}-10^{-14}\ \mathrm{g}^{-1}\,\mathrm{cm}^3\,\mathrm{s}$.

Following the Ericksen-Leslie formalism [6,7], we write down the most general expressions for σ'_{ji} and g'_i

$$\sigma'_{ji}=A_{jik}M_k+A_{jikm}d_{km}, \tag{18}$$

$$g'_i=B_{ij}M_j+B_{ijk}d_{jk}, \tag{19}$$

where $M_i=P_{ik}(\dot{c}_k-\omega_{kj}c_j)$ describes the relative rotation of **c** with respect to the fluid due to torques along **N** and has the properties $M_ic_i=M_iN_i=0$. The most general expression for σ'_{ji} and g'_i consistent with the symmetry of Sm-*C* LCs, [Eq. (14)] and the Onsager reciprocity relations are

$$\sigma'_{ji}=-\frac{\gamma_2}{2}(N_jM_i+N_iM_j)-\frac{\gamma_3}{2}(c_jM_i+c_iM_j)+\frac{\gamma_1}{2}(c_jM_i-c_iM_j)+\alpha_1\delta_{ij}d_{kk}+\alpha_2d_{ij}+\alpha_3(N_kN_md_{km}\delta_{ij}+N_iN_jd_{kk})$$
$$+\alpha_4(N_jN_md_{im}+N_iN_md_{jm})+\alpha_5(c_kc_md_{km}\delta_{ij}+c_ic_jd_{kk})+\alpha_6(c_jc_md_{im}+c_ic_md_{jm})+\frac{\gamma_1}{2}(c_jc_md_{im}-c_ic_md_{jm})$$
$$+\alpha_7(N_kc_md_{km}\delta_{ij}+N_ic_jd_{kk}+N_jc_id_{kk})+\alpha_8(N_jc_md_{im}+N_ic_md_{jm}+N_mc_jd_{im}+N_mc_id_{jm})+\frac{\gamma_2}{2}(N_mc_jd_{im}-N_mc_id_{jm})$$
$$+\alpha_9(N_ic_j+N_jc_i)N_kc_md_{km}+\frac{\gamma_3}{2}(N_jc_i-N_ic_j)N_kc_md_{km}+\alpha_{10}(N_ic_jc_kc_m+N_jc_ic_kc_m+c_ic_jN_kc_m)d_{km}+\alpha_{11}(c_iN_jN_kN_m$$
$$+c_jN_iN_kN_m+N_iN_jc_kN_m)d_{km}+\frac{\gamma_2}{2}(c_iN_j-c_jN_i)N_kN_md_{km}+\alpha_{12}N_iN_jN_kN_md_{km}+\alpha_{13}c_ic_jc_kc_md_{km}, \tag{20}$$

$$g'_i=\gamma_1M_i+\gamma_2N_kd_{ik}+\gamma_3c_kd_{ik}. \tag{21}$$

These expressions can now be compared with those of de Gennes and Prost [12]. In principle, there is full agreement between our approach and that of de Gennes and Prost.

D. Linearized equations

Before we tackle the various situations of macroscopic flow, we will choose a suitable parametrization of the variables and present the hydrodynamic equations in an appropriate form that is convenient to make a more complete comparison with the theory of MPP. Since we will be studying perturbations of a planar structure with the layer normal initially aligned along the Z axis, we choose

$$\psi=z-u(x,y,z). \tag{22}$$

Then a suitable parametrization of the layer normal and the **c** vector is

$$\mathbf{N}\simeq\left(-\frac{\partial u}{\partial x},-\frac{\partial u}{\partial y},1\right), \tag{23}$$

$$\mathbf{c}\simeq\left(\cos\phi,\sin\phi,\frac{\partial u}{\partial x}\cos\phi+\frac{\partial u}{\partial y}\sin\phi\right), \tag{24}$$

where ϕ is the angle made by the **c** vector with the X axis, the direction of the undistorted **c** vector. We take the free energy density up to linear terms in the gradients of u and ignore terms such as $\partial^2u/\partial x_i\partial z$ since the term $\partial u/\partial z$ would be of the highest order. If linearized in ϕ also, the expression reduces to that of de Gennes and Prost.

Finally, the hydrodynamic equations become

$$\frac{D\rho}{Dt}+\rho v_{k,k}=0, \tag{25}$$

$$\rho\frac{Dv_x}{Dt}=-P_{,x}+\sigma'_{jx,j}, \tag{26}$$

$$\rho\frac{Dv_y}{Dt}=-P_{,y}+\sigma'_{jy,j}, \tag{27}$$

$$\rho\frac{Dv_z}{Dt}=-P_{,z}+\sigma'_{jz,j}+\frac{\delta F}{\delta u}, \tag{28}$$

$$\frac{\partial u}{\partial t}-v_z=\lambda_p\frac{\delta F}{\delta u}, \tag{29}$$

$$g'_yc_x-g'_xc_y=\frac{\partial F}{\partial\phi}-\left(\frac{\partial F}{\partial\phi_{,j}}\right)_{,j}, \tag{30}$$

where P $[=\rho^2(\partial F/\partial\rho)]$ is the pressure and

$$\frac{\delta F}{\delta u}=\left(\frac{\partial F}{\partial u_z}\right)_{,z}-\left(\frac{\partial F}{\partial u_{xx}}\right)_{,xx}-\left(\frac{\partial F}{\partial u_{yy}}\right)_{,yy}-2\left(\frac{\partial F}{\partial u_{xy}}\right)_{,xy}.$$

Written in this form, our theory is exactly similar to that of MPP apart from the fact that we have used an *asymmetric* stress tensor. To summarize, the main features of our theory are the following.

(i) We make use of an asymmetric stress tensor and show that our theory does correspond to that of de Gennes and Prost when the hydrodynamic equations are linearized, which was not possible with the theory of LSN.

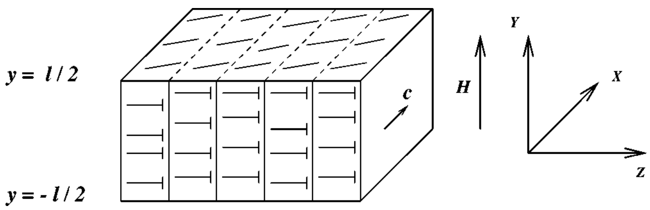

FIG. 1. Schematic representation of the geometry for Fréedericksz transition.

(ii) The theory is covariant in the sense that the constraint $\mathbf{N}\cdot\mathbf{c}=0$ is imposed whereby simultaneous rotations of **N** and **c** do not cost energy. This allows for a general description of large distortions of the **c** vector and the layers. Our theory also allows for variations in layer spacing.

(iii) Permeation is included in the theory.

III. APPLICATIONS

A. Field induced instabilities

In recent times, there has been considerable interest [8–11] in the dynamics of reorientation under the destabilizing influence of a field. The interest has been spurred on by the possibility of fast-switching electrooptic devices. Though we shall be studying the effect of a magnetic field on a sample of aligned Sm-*C* LCs, the results can be extended to the effect of an electric field on surface stabilized ion-free ferroelectric liquid crystals.

We consider the sample to be confined between plates at $y=\pm(l/2)$ with the layers in the so-called bookshelf geometry, i.e., perpendicular to the plates and along the X-Y plane. At the plates, the **c** vector is considered to be rigidly anchored along the X axis. The geometry is depicted in Fig. 1. A magnetic field is applied along the Y direction and acts as a destabilizing field on **c** provided we include the inherent biaxiality of the system and the appropriate diamagnetic anisotropy is positive. This is akin to the familiar splay rich Fréedericksz effect in the homogeneous geometry of a nematic. This instability occurs at a threshold field H_c. When the sample ''switches'' to this state, the **c** vector will develop a component along the Y axis and there will be a transient flow called ''backflow'' in the transverse direction, i.e., the X axis. It has been shown that as in nematic LCs [18,19], here also for $H>H_c$, the effect of backflow is to alter the director profile and the response time of switching during this transient state [8,11]. The linear stability analysis of these authors [8–11], which is similar to that used in uniaxial nematic LCs, assumes that the layers are flat and the existence of flow is along the X direction. Though flow in the direction normal to the layers has also been considered [11], since the effects of permeation have been ignored by Barratt and Duffy, their analysis is incomplete. We have incorporated permeation and reanalyzed the problem. We assume that the hydrodynamic variables are functions of Y and vary with time as e^{ft}, where f is the inverse of a relaxation time. When $f>0$, the perturbation grows and the stationary state becomes unstable. The acceleration terms associated with macroscopic flow may be neglected compared to viscous terms. Since we are interested in the growth of the instability, we shall be concerned with the linearized force and torque equations. The velocity field is taken as $\mathbf{v}=(v_x,0,v_z)$. Then the relevant equations are

$$fa_1\phi_{,y}+a_2v_{x,yy}+a_3v_{z,yy}=0, \tag{31}$$

$$fa_4\phi_{,y}+a_3v_{x,yy}+a_5v_{z,yy}=A_{21}u_{,yyyy}+C_2\phi_{,yyy}, \tag{32}$$

$$fu-v_z=-\lambda_p(A_{21}u_{,yyyy}+C_2\phi_{,yyy}), \tag{33}$$

$$fa_6\phi-a_1v_{x,y}-a_4v_{z,y}=-h\phi-B_2\phi_{,yy}-C_2u_{,yyy}, \tag{34}$$

where $a_1=-(\gamma_1+\gamma_3)/2$, $a_2=\frac{1}{2}(\alpha_2+\alpha_6-\gamma_3-\gamma_1/2)$, $a_3=\frac{1}{4}(2\alpha_8-\gamma_2)$, $a_4=-\gamma_2/2$, $a_5=\frac{1}{2}(\alpha_2+\alpha_4)$, $a_6=\gamma_1$, and $h=\chi_aH^2$. χ_a is the diamagnetic anisotropy associated with the **c** vector. Here and in the rest of the paper we have used the notation $\phi_{,y}=\partial\phi/\partial y$ and $\phi_{,yy}=\partial^2\phi/\partial y^2$.

Equations (31)–(34) have to be solved with the boundary conditions $\phi=v_x=v_z=u=u_{,y}=0$ at the plates. The last condition implies that the layers are clamped at the plates. This condition of layer clamping takes care of one more boundary condition necessary to completely solve the fourth-order differential equation. It should also be noted that if we assume the layers to be flat or the velocity normal to the layers to be zero, we would get an overdetermined set of equations. This is true even if the coupling constant is nonexistent, i.e., $C_2=0$. Hence the earlier results [8,11] obtained under such assumptions are not realistic. Our theory has corrected this error.

1. Effect of C_2

The term in the free-energy density F', that involves C_2 is allowed by the symmetry of the system. It couples distortions in the **c** vector with layer curvature. The equations of equilibrium (33) and (34) couple these variables. We shall now discuss the effects of C_2 on the Freedericksz transition.

The solution that satisfies the equations of equilibrium and the boundary conditions is

$$\phi=A\left(\cosh ky-\cosh\frac{kl}{2}\right), \tag{35}$$

$$u=B\left(\sinh ky-\frac{2y}{d}\sinh\frac{kl}{2}\right)+Cy\left(y^2-\frac{l^2}{4}\right), \tag{36}$$

where $B=-AC_2/A_{21}k$, $C=(2AC_2/A_{21}kl^2)[k\cosh(kl/2)-(2/l)\sinh(kl/2)]$, and the critical field at which the instability sets in is given by the relation $h_c=-(A_{21}B_2-C_2^2)k_c^2/A_{21}$, where k_c is a solution of the transcendental equation $12C_2^2+(A_{21}B_2-C_2^2)(k_cl)^2=[24C_2^2/k_cl]\tanh(k_cl/2)$. It is to be noted that the sign of C_2 does not change the end results and that the threshold is *reduced* because of this coupling parameter. Hence, when distortions in the orientation are coupled to layer curvature, the bare elastic constant B_2 is replaced by a smaller effective elastic constant B_2^{effec}. Another feature of this solution is that if the wave number k_c (k is imaginary) can be extracted experimentally and if it is not equal to πi ($i=\sqrt{-1}$), it will unequivocally establish that $C_2\neq 0$. An indication of layer curvature in the static situation also establishes the same result.

2. Dynamic response

To bring out the main features of the dynamics of reorientation we need not explicitly include C_2 since it only corrects slightly the end results and the form of the solution is not altered. A solution of the differential equations (31)–(34) satisfying the already specified boundary conditions is obtained by assuming the ansatz

$$\phi=\sum_{i=1}^{3}\eta_i\left(\cosh k_i y-\cosh\frac{k_i l}{2}\right),$$

$$v_x=\sum_{i=1}^{3}\mu_i\left(\sinh k_i y-\frac{2y}{l}\sinh\frac{k_i l}{2}\right),$$

$$v_z=\sum_{i=1}^{3}\nu_i\left(\sinh k_i y-\frac{2y}{l}\sinh\frac{k_i l}{2}\right),$$

$$u=\sum_{i=1}^{3}\xi_i\left(\sinh k_i y-\frac{2y}{l}\sinh\frac{k_i l}{2}\right).$$

Then we find that

$$\mu_i=-\frac{f}{k_i X_i}[(a_1a_5-a_3a_4)(f+\lambda_p A_{21}k_i^4)-a_1A_{21}k_i^2]\eta_i, \tag{37}$$

$$\nu_i=-\frac{f}{k_i X_i}[(a_2a_4-a_1a_3)(f+\lambda_p A_{21}k_i^4)]\eta_i, \tag{38}$$

$$\xi_i=-\frac{f}{k_i X_i}(a_2a_4-a_1a_3)\eta_i, \tag{39}$$

$$\eta_2=\frac{X_2}{X_1}\frac{k_2}{k_1}\frac{M_2}{M_1}\eta_1, \tag{40}$$

$$\eta_3=\frac{X_3}{X_1}\frac{k_3}{k_1}\frac{M_3}{M_1}\eta_1, \tag{41}$$

where $X_i=(a_2a_5-a_3^2)(f+\lambda_p A_{21}k_i^4)-a_2A_{21}k_i^2$, $Z_i=k_i\cosh(k_il/2)-(2/d)\sinh(k_il/2)$, $Q_i=(a_1^2a_5-2a_1a_3a_4+a_2a_4^2)(f+\lambda_pA_{21}k_i^4)-a_1^2A_{21}k_i^2$, $M_1=Z_2k_3^4\sinh(k_3l/2)-Z_3k_2^4\sinh(k_2l/2)$, $M_2=Z_3k_1^4\sinh(k_1l/2)-Z_1k_3^4\sinh(k_3l/2)$, $M_3=Z_1k_2^4\sinh(k_2l/2)-Z_2k_1^4\sinh(k_1l/2)$, and the unknowns k_i and f are obtained from

$$X_i(fa_6+h+B_2k_i^2)+fQ_i=0 \tag{42}$$

and

$$\sum_{i=1}^{3}M_i\left[(fa_6+h)k_iX_i\cosh\frac{k_il}{2}+\frac{2f}{l}Q_i\sinh\frac{k_il}{2}\right]=0. \tag{43}$$

Equation (42) is a sixth-order polynomial in k and gives the six roots of the form $\pm k_i$. In the domain of the parameters chosen by us, the roots k_i were either imaginary or real. We have to find that f that fixes these roots as well as satisfies Eq. (43). We make use of the Rayleigh criterion to find the suitable solution, that is, we choose the set of roots with the largest value of f.

With the onset of instability, there exists curvature distortion of the layers and flow along the layer and normal to the layer, that is, along both X and Z axes. Transverse permeation flow normal to the layers is essentially coupled to layer distortions. It can be seen that our results have a different structure when compared with the earlier results [8,11]. There exists analytical solutions to the linear equations, satisfying the boundary conditions with two real values of k and an imaginary k. This is not surprising since the inclusion of layer distortions introduces additional length scales associated with permeation. It should be noted that there are now three wave numbers as compared to one in the previous studies. The previous investigations have noted that the solutions are physical only within a range of values of the parameters. Due to the complexity of Eqs. (42) and (43), we are not able to derive an analytical expression for such a range. Parameters outside this range are not considered since they lead to the unphysical answer of f being positive even for zero field, i.e., the initial bookshelf state would be ''mechanically unstable'' even with no magnetic field present. With the same values of the parameters, the growth rate f deduced by our theory is larger than that of the previous work [8] in all cases, but whether the increase of f is due to the additional resistance provided by layer curvature or it is just a consequence of the extra viscosity coefficients that exist in our general theory is not clear.

The other important consequences of our analysis are the following

(i) We note that the distortion of the **c** vector has similar features to those as found in earlier studies [8]. That is, the effects of backflow become important at higher values of the field and the variation of the backflow effects with a particular ratio of the material parameters is as in earlier works [8,11]. Similarly, the imaginary k saturates with the field at a particular value. This is surprising in view of the fact that the governing equations (42) and (43) are very different in the present problem. With a strong field, the ϕ profile shows the interesting feature of having an opposite twist near the plates and near the center. The profiles of the ϕ distortions for two different h $(=\chi_aH^2)$ values are shown in Figs. 2(a) and 3(a).

(ii) It has been generally assumed in previous investigations that the flow in the direction normal to the layers is negligible when compared to that parallel to the layers, due to a large viscosity in that direction. This is a reasonable approximation at low fields. Even our calculations of scaled velocities as shown in Figs. 2(b) and 2(c) show this. However, we find that as the field is *increased*, the transient flow velocity in the two directions can become *comparable*. This is apparent in the scaled velocities shown in Figs. 3(b) and 3(c). It should be remembered that the amplitude of the profiles cannot be extracted from this theory just as the amplitude of distortion in a Fréedericksz transition is not obtainable from a linear theory. It is to be mentioned that our calculations are for certain assumed values for the parameters since experimental values are still lacking. Apart from this, it should also be noted that the symmetry of the equations does not allow the flow along the layer normal to be

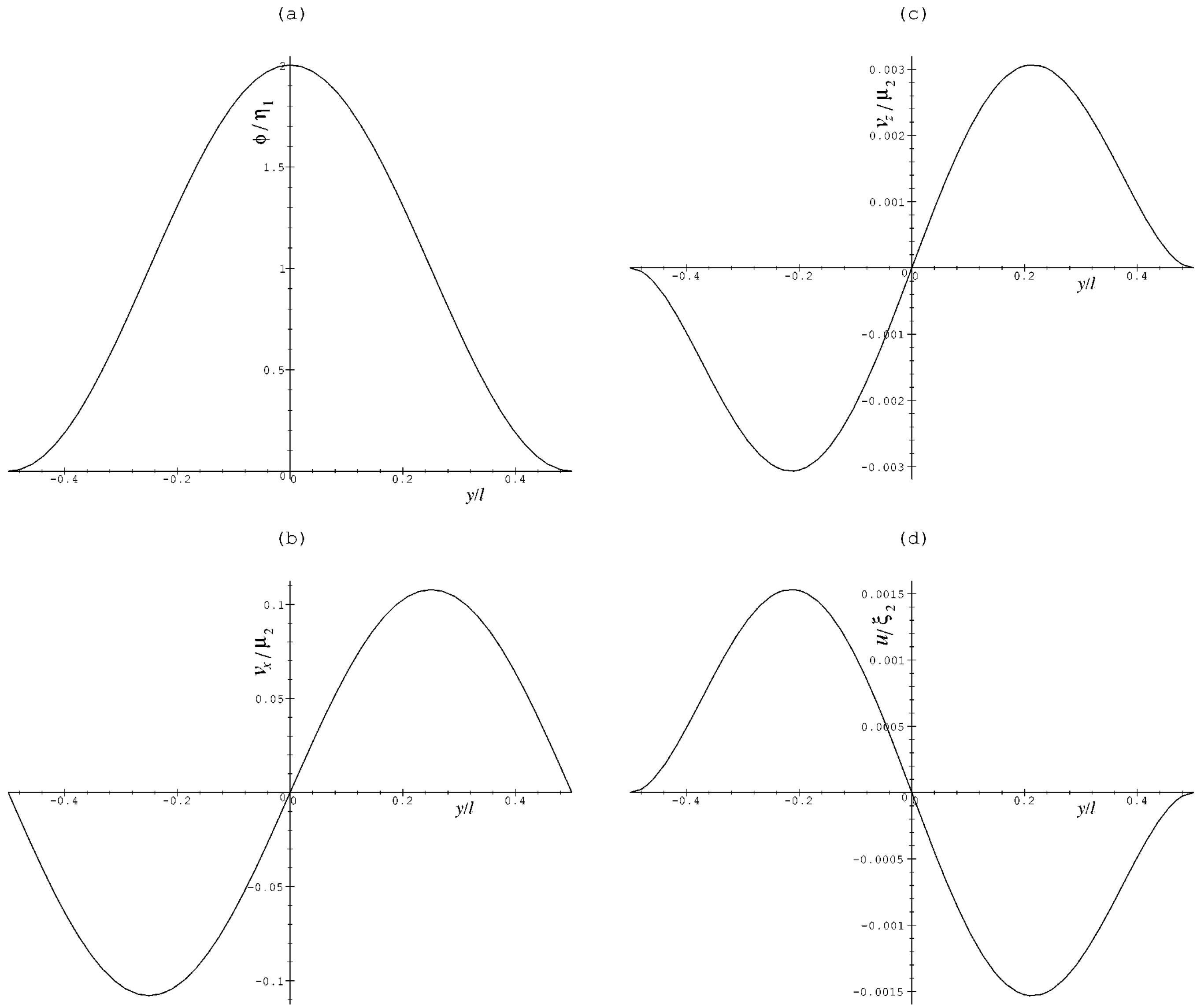

FIG. 2. The threshold h_c for the instability is $h_c \simeq 10$ ergs cm^{-3}. Here $h=50$ ergs cm^{-3}. The separation between the plates is $d=10^{-3}$ cm. The parameters are $\lambda_p=10^{-14}$ g^{-1} cm^3 sec, $A_{21}=3\times10^{-6}$ dyn, $B_2=10^{-6}$ dyn, $a_1=2$ P, $a_2=5$ P, $a_3=2.3419$ P, $a_4=1$ P, $a_5=1.1169$ P, and $a_6=-1.01$ P. At this value of the field, it can be seen that the ϕ profile is hardly affected by the ''backflow.'' The layer distortion is also small. The velocity normal to the layers is much smaller than that along the layers. (a) ϕ, (b) v_x, (c) v_z, and (d) u. Here ϕ, v_x, v_z and u have been normalized by η_1, μ_2, μ_2, and ξ_2, respectively, to give dimensionless quantities. In this case, $f=49.5760$ sec^{-1}, and $k_1 d = \pm 6.2985i$, $k_2 d = \pm 0.6422$, and $k_3 d = \pm 122\,474.4871$, where $i=\sqrt{-1}$.

plug flow. It can be seen that if ϕ is an even function of y, then the other quantities v_x, v_z, and u will be odd functions and these features are clearly shown in Figs. 2 and 3. However, for practical situations, the other permitted solution with odd ϕ and even v_x, v_z, and u are improbable.

B. Flow induced instabilities

It has been shown [4,12] that in the shear flow with one plate moving relative to the other, the stationary state, when the layers are parallel to the plates, has **c** either parallel or antiparallel to the direction of main velocity provided there is no surface anchoring of **c**. Further, which direction **c** adopts depends on whether the product of the shear rate and a certain viscosity coefficient is positive or negative. On the other hand, when the layers are perpendicular to the plates and the shear plane is parallel to the layers, we get a different stationary state. Here **c** is aligned everywhere at a particular angle with respect to the flow direction provided **c** is not anchored at the walls. This angle is called the *Leslie angle* in the case of uniaxial nematic LCs. This state occurs only when the ratio of certain viscosity coefficients is less than one; otherwise the phenomenon of tumbling occurs where **c** continuously rotates as we go from one plate to the other. We consider here these geometries, but with surface anchoring thus bringing in elastic distortions in the **c** vector and consequently layers.

1. Shear plane normal to the layers

We consider the layers to be parallel to the plates. The **c** vector is aligned everywhere along the Y axis, the direction of shear flow. This state is described by $\mathbf{c}=(0,1,0)$ and $\mathbf{v}=(0,2sz,0)$. The geometry is shown in Fig. 4. The hydrody-

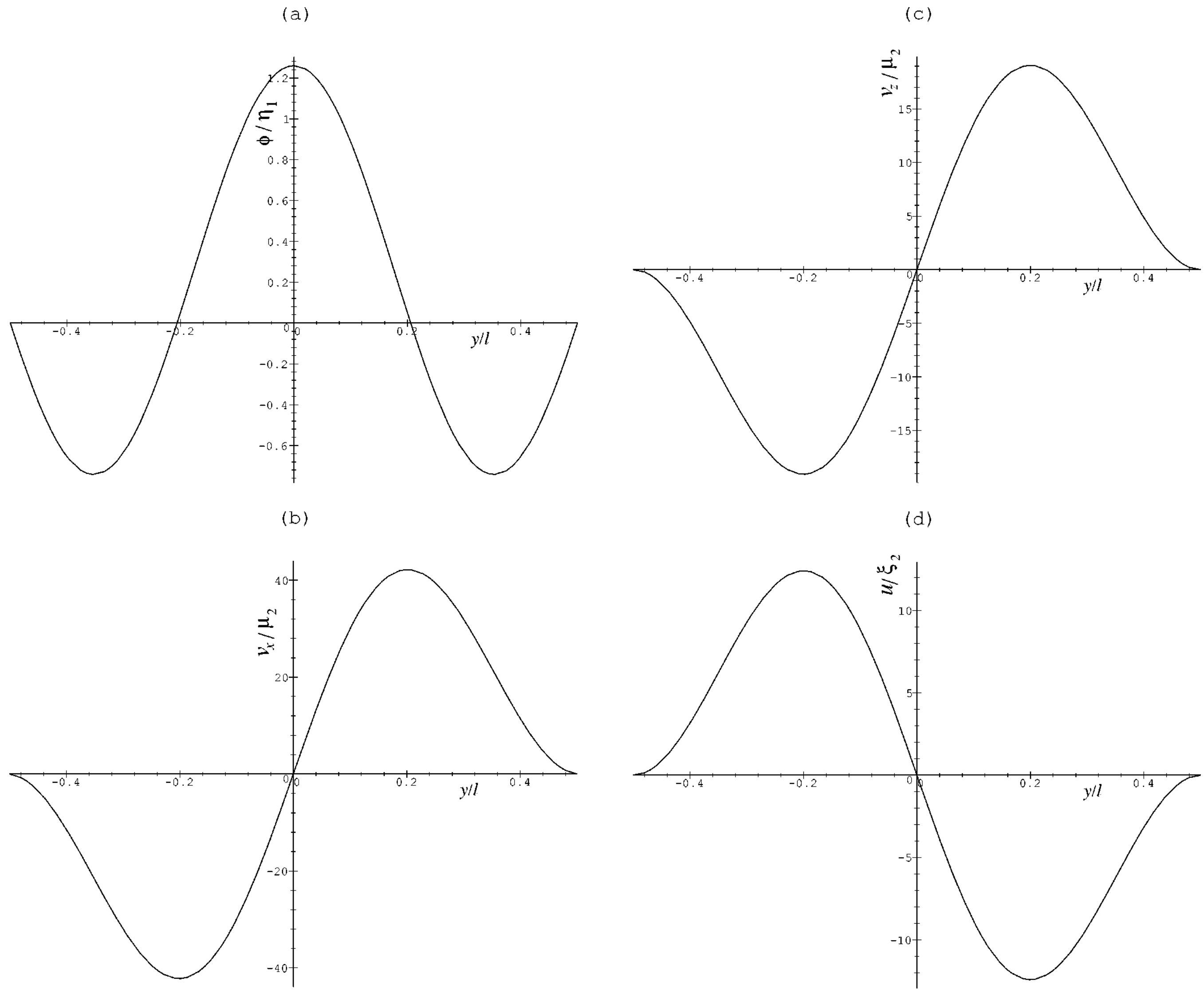

FIG. 3. $h = 200$ ergs cm^{-3}, and the rest of the parameters are the same as in Fig. 2. (a) ϕ, (b) v_x, (c) v_z, and (d) u. It can be seen in (a) that the orientation of the **c** vector at the walls is opposite to that within the bulk. From (b) and (c) it can be seen that the velocities are comparable. The normalization of the quantities is as described in Fig. 2. In this case, $f = 602.3026\ \mathrm{sec}^{-1}$, $k_1 d = \pm 8.9062i$, $k_2 d = \pm 3.1336$, and $k_3 d = \pm 122\,474.4866$, where $i = \sqrt{-1}$.

namic equations indicate that there exists an instability in this case. To describe the onset of the instability, we undertake a linear stability analysis. We assume $\mathbf{v} = (2v_x(z), 2sz, 0)$ and $\mathbf{c} \simeq (\phi(z), 1, 0)$. The hydrodynamic equations do not lead to either layer distortion or permeation flow. As usual, we assume a e^{ft} time dependence. Then the relevant equations are

$$(fb_1 - b_2 s)\phi_{,z} + b_3 v_{x,zz} = 0, \tag{44}$$

$$B_3 \phi_{,zz} - (b_1 s - f b_4)\phi - b_1 v_{x,z} = 0, \tag{45}$$

where $b_1 = \gamma_2$, $b_2 = \alpha_6 + \alpha_9$, $b_3 = \alpha_2 + \alpha_4$, and $b_4 = \gamma_1$. We impose a no-slip condition for **v** and rigid anchoring for **c** at the boundaries. The solution takes the form

$$\phi = A\left[\cos kz - \cos k\frac{l}{2}\right], \tag{46}$$

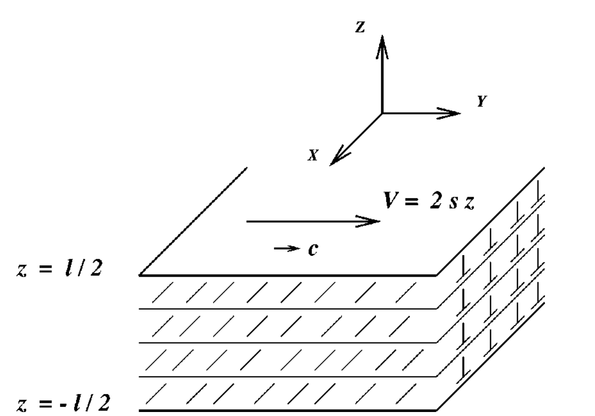

FIG. 4. Schematic representation of the geometry with shear flow of the layers that are parallel to the walls and are sliding past each other. The shear plane is perpendicular to the layers. The nail representation is used and the nail head indicates that the **c** vector goes into the plane of the paper.

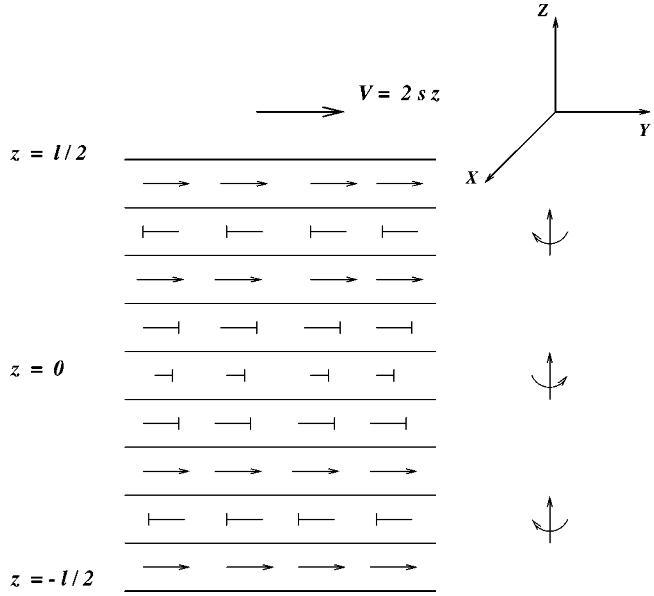

FIG. 5. Schematic representation of the distortion in **c** associated with the geometry described in Fig. 4. It can be seen that the twist distortion near the walls is opposite in a sense to the twist distortion near the center.

$$v_x = B\left[\sin kz - \frac{2z}{l}\sin k\frac{l}{2}\right], \tag{47}$$

with $B = -A(\lambda b_1 - b_2 k)/b_3 k$. The values of k and f are obtained from

$$sb_1(b_1^2 - b_2 b_4)\left[1 - \frac{2}{kl}\tan k\frac{l}{2}\right] = B_3 k^2\left[b_3 b_4 + \frac{2}{kl}b_1^2 \tan k\frac{l}{2}\right], \tag{48}$$

$$f(b_3 b_4 + b_1^2) = b_3\left[B_3 k^2 + b_1 s\left(1 + \frac{b_2}{b_3}\right)\right]. \tag{49}$$

For $s > 0$ $(s < 0)$, there will be an instability above a threshold value only if $b_1 = \gamma_2 < 0$ $(b_1 > 0)$.

Above this threshold shear there exists a *transverse flow* v_x, but unlike in the Fréedericksz transition, this is *not a transient phenomenon*. Also, well above the threshold, the twist distortion of the **c** vector near the center is opposite to the twist near the plates. A schematic representation of such a **c** vector distortion is depicted in Fig. 5. When the **c** vector is aligned everywhere along the X axis with an imposed shear flow in the Y direction, there is no threshold for instability.

2. Shear plane parallel to the layers

In this case, the layers are normal to the plates in the bookshelf geometry as shown in Fig. 6 and the shear plane is parallel to the layers $\mathbf{v} = (2sy, 0, 0)$ and $\mathbf{c} = (\cos\phi, \sin\phi, 0)$. With $|\gamma_1/\gamma_3| < 1$ and no **c** vector anchoring at the walls, we can readily verify that the continuum equations lead to a shear-independent **c** vector orientation given by

$$\cos 2\phi_0 = -\frac{\gamma_1}{\gamma_3}. \tag{50}$$

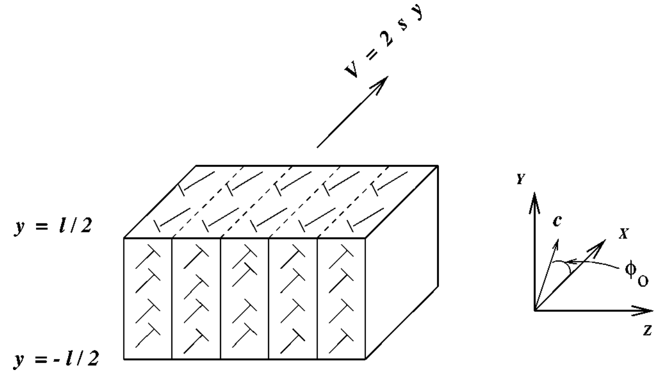

FIG. 6. Schematic representation of the geometry with the shear flow along the layers that are perpendicular to the walls. The shear plane is parallel to the layers. Here we show the undistorted sample with the wall anchoring of $\mathbf{c} = (\cos\phi_0, \sin\phi_0, 0)$.

In this flow, the layers are flat and parallel to X-Y plane. It is easy to see that this simple solution is realizable with the same surface alignment of $\mathbf{c} = (\cos\phi_0, \sin\phi_0, 0)$ at the walls. Hence, at the boundary and elsewhere, the **c** orientation is $\mathbf{c} = (\cos\phi_0, \sin\phi_0, 0)$. Here we examine the stability of this solution. We shall assume spatial variations only along the Y direction. The analysis is similar to that in field induced instabilities. We take $\mathbf{v} = (2sy + 2v_x, 0, 2v_z)$ and $\mathbf{c} = (\cos(\phi_0 + \phi), \sin(\phi_0 + \phi), 0)$. Once again the instability involves a distortion in the **c** vector and also distortion of the layers due to transverse permeation flow given by v_z. It should be noted that the velocity along the direction of shear will not be linear, but corrected by a nonlinear function. To estimate the threshold shear rate s_c we assume that the perturbations have an e^{ft} time dependence. The instability is assumed to be stationary whereby the principle of exchange of stabilities is valid and $f = 0$ at threshold [20]. The linearized hydrodynamic equations are

$$sc_1 u_{,yy} + c_2 v_{x,yy} + sc_3\phi_{,y} + c_4 v_{z,yy} = 0, \tag{51}$$

$$sc_5 u_{,yy} + c_6 v_{x,yy} + sc_7\phi_{,y} + c_8 v_{z,yy} + c_9 u_{,yyyy} = 0, \tag{52}$$

$$-2v_z = \lambda_p c_9 u_{,yyyy}, \tag{53}$$

$$sc_{10} u_{,y} + sc_{11}\phi + c_{12} v_{z,y} + c_{13}\phi_{,yy} = 0. \tag{54}$$

Taking $\varphi_1 = \sin\phi_0$ and $\varphi_2 = \cos\phi_0$, $c_1 = -\gamma_2\varphi_1 - 2\alpha_8\varphi_1 - 3\alpha_{10}\varphi_1\varphi_2^2$, $c_2 = -\gamma_3(\varphi_2^2 - \varphi_1^2) - \gamma_1/2 + \alpha_2 + \alpha_6 + 2\alpha_{13}\varphi_1^2\varphi_2^2$, $c_3 = 4\gamma_3\varphi_1\varphi_2 + 4\alpha_{13}\varphi_1\varphi_2(\varphi_2^2 - \varphi_1^2)$, $c_4 = \alpha_8\varphi_2 - (\gamma_2/2)\varphi_2 + \alpha_{10}\varphi_2\varphi_1^2$, $c_5 = -(\gamma_3 - \alpha_6 + \alpha_9 - 2\alpha_{13}\varphi_1^2)\varphi_1\varphi_2$, $c_6 = -(\gamma_2/2)\varphi_2$, $c_7 = (\gamma_2/2)\varphi_1 - \alpha_8\varphi_1 + 2\alpha_{10}(2\varphi_1\varphi_2^2 - \varphi_1^3)$, $c_8 = \alpha_2 + \alpha_4 + (\alpha_6 + \alpha_9)\varphi_1^2$, $c_9 = -(A_{11}\varphi_1^2\varphi_2^2 + A_{21}\varphi_2^4 + A_{12}\varphi_1^4)$, $c_{10} = \gamma_2\varphi_1$, $c_{11} = -4\gamma_3\varphi_1\varphi_2$, $c_{12} = \gamma_2\varphi_2$, and $c_{13} = B_1\varphi_1^2 + B_2\varphi_2^2$.

Here we have neglected the elastic coupling between distortions in the **c** vector and those of the layers, i.e., $C_2 = 0$. These equations are similar to Eqs. (31)–(34) and the solutions for v_x, v_z, ϕ, and u are of the same form. The two relations that yield the threshold shear are

$$s(c_2c_7-c_3c_6)[sc_{10}-\tfrac{1}{2}\lambda_p c_9 c_{12}k_i^4]-(sc_{11}+c_{13}k_i^2)[s(c_2c_5-c_1c_6)+c_2c_9k_i^2-\tfrac{1}{2}\lambda_p c_9(c_2c_8-c_4c_6)k_i^4]=0, \quad (55)$$

$$\sum_{i=1}^{3} P_iQ_iR_i=0, \quad (56)$$

where $P_i=sc_{10}-\frac{1}{2}i\lambda_p c_9c_{12}k_i^4$, $R_i=(2/l)\sinh\frac{1}{2}k_il-[sc_{11}/(sc_{11}+c_{13}k_i^2)]k_i\cosh\frac{1}{2}k_il$, $Q_1=Z_2k_3^4\sinh\frac{1}{2}k_3l-Z_3k_2^4\sinh\frac{1}{2}k_2l$, $Q_2=Z_3k_1^4\sinh\frac{1}{2}k_1l-Z_1k_3^4\sinh\frac{1}{2}k_3l$, $Q_3=Z_1k_2^4\sinh\frac{1}{2}k_2l-Z_2k_1^4\sinh\frac{1}{2}k_1l$, with $Z_i=k_i\cosh\frac{1}{2}k_il-(2/l)\sinh\frac{1}{2}k_il$. From these equations we can show that the threshold shear rate varies inversely as the square of the sample thickness and *increases* with increasing λ_p. The latter implies that as the viscosity *reduces* in the Z direction, a larger shear rate is required to make the stationary state unstable. It should be emphasized that here also the transverse permeation flow and layer curvature are *not transient* in this flow induced instability.

C. Poiseuille flow

1. Inlet section

In the classical Poiseuille flow, liquid flows into a capillary or the space between two parallel plates from a pressure head. The liquid enters with nearly a flat profile. A boundary layer starts to develop at the walls at the entry point. As the liquid flows down, the boundary layer thickness continuously increases. A point downstream will be reached where the boundary layers with their nearly parabolic velocity profiles meet or crossover at the center. It is only after this point the velocity profile is stationary and becomes symmetrically parabolic. Over the distance from the entry point to the crossover point, called the *inlet section*, the velocity profile over the cross section continuously changes as we go down the tube. Hence, in the inlet section, the liquid in the central region is continuosly accelerated and that at the walls is always at rest.

We present a simple analysis to estimate the length of this inlet section. We consider the case where the smectic layers are parallel to the plates that are in the X-Y plane. Further, the velocity field is described by $\mathbf{v}=(v_x,0,v_z)$. The following equations govern the flows v_x and v_z [12]:

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_z}{\partial z}=0, \quad (57)$$

$$\frac{\partial p}{\partial z}=-\frac{v_z}{\lambda_p}, \quad (58)$$

$$\rho\left(v_x\frac{\partial v_x}{\partial x}+v_z\frac{\partial v_x}{\partial z}\right)=-\frac{\partial p}{\partial x}+\eta\frac{\partial^2 v_x}{\partial z^2}. \quad (59)$$

Here η is the effective coefficient of viscosity and λ_p is the permeation constant. We can estimate from (57)–(59) the boundary layer thickness δ up to which the transverse pressure gradient extends. We find δ to be a function of the distance x from the edge of the plate. Let the flow at the center be V_0. Then, from the continuity equation, a rough estimate gives $v_z=V_0\delta/x$. Then, from Eq. (58) we get the pressure p, $p=V_0\delta^2/x\lambda_p$. Equation (59) simplifies to $x^2-2r\delta x-\delta^4/\kappa^2=0$, where $r=\delta V_0\rho/\eta$ and $\kappa=\sqrt{\lambda_p\eta}$, the length scale introduced by permeation effects. Thus we get for the length of the inlet section L,

$$L=l\left[R+\frac{l}{\kappa}\sqrt{1+\rho^2V_0^2\frac{\kappa^2}{\eta^2}}\right], \quad (60)$$

where $R=\rho V_0 l/\eta$ is the Reynold's number and l the separation between the plates. From this it can be seen that we get the length of inlet section as $2lR$ for ordinary fluids in the limit $\kappa\to\infty$. Since viscosity dominates flows at small l and V_0, this inlet section is comparable to l and depends on main flow velocity V_0. In smectic LCs, κ is very small, being of the order of the layer thickness. Then the inlet section is given by

$$L\simeq\frac{l^2}{\kappa}. \quad (61)$$

This is invariably *very large* compared to l and is independent of the main flow velocity V_0. If $l\sim10^{-3}$ cm, $L\sim10$ cm. Hence actual experimental results in short tubes are not really representative of the steady Poisseuille flow. This result is true of all smectic LCs in general. In this context, it is relevant to recall here a recent calculation by Walton, Stewart, and Towler [21] on the flow past finite obstacles in smectic liquid crystals. This calculation is an application of the theory of hydrodynamics of smectic-*A* LCs, i.e., it explicitly incorporates the permeation process, which is an essential feature of flow past obstacles.

2. Hall effect

In the case of uniaxial nematic LCs, it is known that there exists in a planar Poiseuille flow a transverse pressure gradient when the director is held at an angle, say, by an infinitely strong magnetic field, with respect to plane of shear. In this section we shall briefly describe an analogous situation in Sm-*C* LCs.

We consider the geometry of Fig. 6. The state $\mathbf{N}=(0,0,1)$, $\mathbf{c}=(\cos\phi,\sin\phi,0)$, and $\mathbf{v}=(2v_x(y),0,0)$ is stable below a threshold shear if we assume that a strong magnetic field aligns $\mathbf{c}$ at a particular angle ϕ. Then

$$P_{,x}=\frac{d^2v_x}{dy^2}\left[\alpha_2+\alpha_6+2\alpha_{13}\cos^2\phi\sin^2\phi-\frac{\gamma_1}{2}-\gamma_3\cos 2\phi\right], \quad (62)$$

$$P_{,z}=\cos\phi\frac{d^2v_x}{dy^2}\left[\alpha_8-\frac{\gamma_2}{2}+2\alpha_{10}\sin^2\phi\right]. \quad (63)$$

It can be seen that there exists a transverse pressure gradient $P_{,z}$ whenever the $\mathbf{c}$ vector is not along Y, i.e., $\phi=\frac{1}{2}\pi$, or at a particular angle $\phi=\sin^{-1}[\sqrt{(\gamma_2-2\alpha_8)/4\alpha_{10}}]$. The additional feature compared to nematic LCs is that there are three possibilities in view of Eqs. (28) and (29): (i) A pressure gradient develops along the Z direction without any lattice dilatation or flow along Z, (ii) the pressure gradient is accompanied by layer dilatation, but without any macroscopic flow along the layer normal, and (iii) the pressure gradient leads to layer dilatation and flow along the layer normal.

In the absence of an external field also, the vector will align, but along the Leslie angle ϕ_0. Hence the transverse pressure gradient exists even in such a situation, unless ϕ_0 actually happens to be equal to $\frac{1}{2}\pi$ or $\sin^{-1}[\sqrt{(\gamma_2-2\alpha_8)/4\alpha_{10}}]$. This type of Hall effect is peculiar to Sm-C LCs.

IV. CONCLUSION

We have developed a macroscopic hydrodynamic theory of Sm-C liquid crystals. This is described by the hydrodynamic variables, viz., the material density, the fluid velocity, the energy density, the scalar variable that describes the layering, and the vector **c**. The asymmetric stress tensor consists of 16 shear viscosity coefficients, three of which are associated with dissipative torques. Our theory agrees with the earlier theories [1,12] where a symmetric stress tensor has been assumed. In the study of the reorientation dynamics of the director in an external field, we showed that assuming the layers to be flat or the velocity normal to the layers to be zero leads to an overdetermined set of equations. There exists coupling between curvature of the layers and orientation of the director that alters the director profile. In the transient stage, layer distortion can arise even in the absence of the coupling parameter. The velocity normal to the layers can become comparable to that along the layers. Then, we considered **c** vector instabilities in shear flows. When there is a shear flow with the layers slipping past each other and the flow is along the direction of **c**, there is a threshold shear above which the uniformly aligned Sm-C LC becomes unstable. This leads to a transverse flow along the layers. The twist distortion of the **c** vector near the center is opposite to the twist near the plates. When the shear is along the layers, there develops again above a threshold shear an instability of the uniform sample with **c** vector at the Leslie angle. This instability has transverse permeation flow and layer curvature. Even below these threshold shears, these Poiseuille flows are associated with some special features. In the case of layers sliding past each other, the length of the inlet section is very large compared to the sample thickness and is independent of main flow velocity. In the case of flow and shear parallel to the layers, a transverse pressure gradient exists even in the absence of an aligning field and this is peculiar to Sm-C LCs.

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